Infinitary Lambda Calculus and Discrimination of Berarducci Trees [1]

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We propose an extension of lambda calculus for which the Berarducci trees equality coincides with observational equivalence, when we observe rootstable or rootactive behavior of terms. In one direction the proof is an adaptation of the classical Böhm out technique. In the other direction the proof is based on confluence for strongly converging reductions in this extension.

Key words: Berarducci trees, observational equivalence, infinitary lambda calculus, Böhm out technique.

1 Introduction

In this paper we will prove equivalent an operational and a denotational semantics for lambda calculus with the β-rule. Both semantics are based on the

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set of rootactive terms, which is the smallest set of computational meaningless terms that can consistently be equated. The operational semantics that we are interested in is observational equivalence with respect to rootactive behavior. The denotational semantics is the model of the Berarducci trees (Berarducci, 1996), which are a more detailed variant of Böhm trees: the main difference being that rootactive terms instead of terms without a head normal form get replaced by a dummy symbol $\perp$.

Berarducci trees can be best dealt with as the β-$\perp$-normal forms of terms in the completion $\Lambda^\infty_\perp$ of the lambda calculus with a new symbol $\perp$, infinite terms and the new rule that replaces β-rootactive terms by $\perp$ (Berarducci, 1996), (Kennaway et al., 1997). In the calculus $\Lambda^\infty_\perp$ it is not difficult to see that if two terms have the same Berarducci tree then they are observationally equivalent. The converse however is not true, essentially for two reasons. The first reason is intrinsic for Berarducci trees: the lambda calculus is not powerful enough to Böhm out the argument of a β-rootactive term. The second reason is similar to why the analogous statement is not true for Böhm trees, namely the Böhm out of a subterm can return an η-expansion of it.

Therefore, in a move characteristic for full abstractness problems, we will enrich the lambda calculus $\Lambda^\infty_\perp$ in a rather minimal way with two more constants O and A with accompanying rules. Any term in the enriched calculus $\Lambda^\infty_{\perp\mathsf{OA}}$ can reduce in a finite number of steps to a pure λ-term and therefore the Berarducci tree of a term in $\Lambda^\infty_{\perp\mathsf{OA}}$ will not contain O and A. For any two terms of $\Lambda^\infty_{\perp\mathsf{OA}}$ we can prove that Berarducci tree equality is equivalent to observational equivalence with respect to rootactive terms.

1.1 Previous Work

Historically, quoting from (Barendregt, 1984)(page 215), "the notion of Böhm tree is suggested by the original proof of Böhm's theorem". Böhm's theorem states that given two distinct β-η-normal forms there is a context $C[\ ]$ such that $C[M] = x$ and $C[N] = y$, where x, y are arbitrary distinct variables. The method used to find such a context is called the *Böhm out* technique (Barendregt, 1984)(Section 10.3).

In (Wadsworth, 1976) Wadsworth, generalizing Böhm's theorem, shows that two λ-terms M, N have the same Böhm tree modulo infinite η-expansions if and only if for all contexts $C[\ ]$ the following holds:

$$C[M] \text{ has a head normal form} \quad \Leftrightarrow \quad C[N] \text{ has a head normal form.}$$

The proof technique used to obtain the "if" part is the Böhm out technique.

The same property holds even considering Böhm trees modulo finite η-expansions and normal forms, as shown in (Hyland, 1975/76). More precisely Hyland proves, using the Böhm out technique, that two λ-terms M, N have the same Böhm tree modulo finite η-expansions if and only if for all contexts $C[\,]$ the following holds:

$$C[M] \text{ has a normal form} \quad \Leftrightarrow \quad C[N] \text{ has a normal form.}$$

The results of (Wadsworth, 1976) and (Hyland, 1975/76) can be rephrased as follows:
The lambda calculus internally discriminates as Böhm tree modulo infinite (respectively finite) η-expansions when the set of values is the set of head normal forms (respectively normal forms).
To internally discriminate terms having different Böhm trees Dezani et al. (Dezani-Ciancaglini et al., 1998b) add to the pure lambda calculus a *non-deterministic choice* operator $+$ and an *adequate numeral system* (as defined in Section 6.4 of (Barendregt, 1984)). The reduction rules for $+$ are:

$$M + N \longrightarrow M \quad \text{and} \quad M + N \longrightarrow N.$$

Clearly the non-deterministic choice operator allows to define combinators like Plotkin's parallel-or (Plotkin, 1977) when one considers *may convergence*, under which a term converges if at least one of the possible computations starting from it ends. This extension increases the power of the lambda calculus to detect convergence internally also in those cases in which a term converges as soon as at least one of its subterms does, no matter in which order they are evaluated. This amounts to have the definability of all compact points in a standard model, that is, by Milner's theorem (Milner, 1977), to have a fully abstract interpretation for the language. The numerals play an essential role to discriminate between a term possessing a head normal form and its η-expansion, essentially since they can never be applied to an argument, while all pure λ-terms can be seen both as functions and as arguments. This result is proved using a variation of the Böhm out technique as well as *characteristic terms* and *test terms* (Boudol, 1994).

Instead, Lévy-Longo trees correspond to observational equivalence with respect to weak head normal forms in suitably enriched versions of the lambda calculus, as shown in (Sangiorgi, 1994), (Boudol and Laneve, 1996), (Dezani-Ciancaglini et al., 1999). Now, we briefly recall such approaches.

In (Sangiorgi, 1994), Sangiorgi considers the embedding of lazy lambda calculus in some concurrent calculi. First, Milner's encoding of lazy lambda calculus in π-*calculus* is studied. Then the lazy lambda calculus is enriched with a simple non-deterministic operator, which, when applied to an argument, either gives the argument itself or diverges. In both cases the processes are compared using bisimulation. The proof technique is the Böhm out technique.

Boudol and Laneve (Boudol and Laneve, 1996) introduce a "resource conscious" refinement of lambda calculus, in which every argument comes with a *multiplicity*. The reduction process (which uses *explicit substitutions* in an essential way) remains deterministic, but a *deadlock* can appear. The terms are compared by means of the standard observational equivalence. The proof technique is again the Böhm out technique.

Dezani et al. (Dezani-Ciancaglini et al., 1999) consider the behavior of pure λ-terms inside contexts of the concurrent lambda calculus as defined in (Dezani-Ciancaglini et al., 1998a). This calculus is obtained from the pure lambda calculus (with call-by-value and call-by-name variables) by adding the non-deterministic choice operator discussed above and a parallel operator $\|$, whose main reduction rule is

$$\frac{M \longrightarrow M' \quad N \longrightarrow N'}{M\|N \longrightarrow M'\|N'}\,(\|)$$

where $\longrightarrow$ stands for one-step reduction.

The terms are compared by means of the standard observational equivalence. The proof technique for proving that observational equivalence implies tree equality is that of characteristic terms and test terms.

More recently Boudol in (Boudol, 2000) shows that the equivalence on λ-terms induced by the call-by-name CSP transform is Lévy-Longo tree equality.

In order to discriminate pure λ-terms having different Berarducci trees, the paper (Dezani-Ciancaglini et al., 2000) extends the lambda calculus with two constants O and A. The essential feature of the Böhm-out technique consists in selecting a subtree of the tree of a term by means of an appropriate context. The selection of a subtree was performed in the original Böhm algorithm by substituting a variable in head position by an appropriate combinator. For Berarducci trees, the top normal forms also include applications that may not have a variable in head position, as in $\Omega\Omega$ (where $\Omega \equiv (\lambda x.xx)(\lambda x.xx)$). For these new cases the selection of a subtree can be performed using the constants O and A. The constants O and A select the operator and the argument of a closed, β-rootstable application. These constants have the following reduction rules:

$$\mathsf{O}(MN) \longrightarrow M \text{ if } M \text{ is a closed } \beta\text{-zero term}$$
$$\mathsf{A}(MN) \longrightarrow N \text{ if } M \text{ is a closed } \beta\text{-zero term}$$

where a β-zero term is defined in Definition 2. For instance, $\Omega\mathbf{II}$ (where $\mathbf{I} \equiv \lambda x.x$) and $\Omega\Omega\mathbf{I}$ are discriminated by the context $\mathsf{A}(\mathsf{O}[\,])$. In fact:

$$\mathsf{A}(\mathsf{O}(\Omega\mathbf{II})) \longrightarrow \mathsf{A}(\Omega\mathbf{I}) \longrightarrow \mathbf{I}$$
$$\mathsf{A}(\mathsf{O}(\Omega\Omega\mathbf{I})) \longrightarrow \mathsf{A}(\Omega\Omega) \longrightarrow \Omega$$

All pure λ-terms having different Berarducci trees can be discriminated using these two constants (Dezani-Ciancaglini et al., 2000). However non-pure λ-terms having different Berarducci trees cannot be discriminated only with these rules. For example, $\mathsf{O}(\mathsf{A}\Omega)$ and $\mathsf{O}(\mathsf{O}\Omega)$ have different Berarducci trees, though they are observationally equivalent. Hence in this paper we add more reduction rules for the constants O and A in order to discriminate also non-pure λ-terms.

1.2 Summary

In this paper we consider an extended lambda calculus Λ_{OA} for which the equality of Berarducci trees coincides with observational equivalence. This calculus will be a variant of the one presented in (Dezani-Ciancaglini et al., 2000). As in (Dezani-Ciancaglini et al., 2000) it will contain the constants O and A that select the operator and argument of a β-rootstable application. The set Λ_{OA} of terms will be a restriction of the one in (Dezani-Ciancaglini et al., 2000) and new reduction rules will be added for the constants. In (Dezani-Ciancaglini et al., 2000), we have proved that Berarducci tree equality coincides with observational equivalence only for pure λ-terms. The new reduction rules will allow us to extend this result to non-pure λ-terms. Hence in this paper, we will prove:

Theorem 1 *For all $X, Y \in \Lambda_{\mathsf{OA}}$ it holds that they have the same Berarducci tree if and only if for all contexts $C[\,] \in \Lambda_{\mathsf{OA}}$*

$$C[X] \in R_{\mathsf{OA}} \Leftrightarrow C[Y] \in R_{\mathsf{OA}}$$

where R_{OA} is the set of $\beta\mathsf{OA}$-rootactive terms in Λ_{OA}. [4]

The "if" part will be proved by a variation on the Böhm out technique. For the "only if" part we adapt techniques from infinitary lambda calculus. We will prove that the Berarducci tree of a term is the unique normal form of the term in that calculus. Since this normal form always exists and is unique, we can build a model of the extended lambda calculus in which the interpretations of

[4] The definition of $\beta\mathsf{OA}$-rootactive is given in Definition 27.

terms are their Berarducci trees. Hence, our main theorem states that such a model of the extended lambda calculus is fully abstract.

1.3 Outline

In Section 2 we recall the definition of the finite lambda calculus Λ and its infinitary extension $\Lambda^{\infty}_{\perp}$. We explain that the Berarducci tree of a term M in $\Lambda^{\infty}_{\perp}$ is just its normal form in $\Lambda^{\infty}_{\perp}$. However nice the properties of $\Lambda^{\infty}_{\perp}$, it is not expressive enough to prove that observational equivalence implies Berarducci tree equality. Therefore we introduce in Section 3 the infinitary extension $\Lambda^{\infty}_{\perp\mathsf{OA}}$. It is more expressive than $\Lambda^{\infty}_{\perp}$, but inherits some of its nice properties. In Section 4 we show for terms in $\Lambda^{\infty}_{\perp\mathsf{OA}}$ that Berarducci tree equality implies observational equivalence, and in Section 5 we prove the converse. The final Section 6 discusses the result.

2 Finite and Infinite Lambda Calculus

This section is to fix notations and concepts. We will recall the infinitary extension $\Lambda^{\infty}_{\perp}$ of the finite lambda calculus (Berarducci, 1996), (Kennaway et al., 1997). This is an extension not only with infinite terms but also with an extra symbol $\perp$ and a rewrite rule

$$\frac{M \neq \perp \text{ and } \beta\text{-rootactive}}{M \to \perp}\,(\perp)$$

where β-rootactivity is defined in Definition 4(ii).

The extension $\Lambda^{\infty}_{\perp}$ has the following important properties:

- the infinitary confluence property holds,
- each term has a unique normal form for the combined $\beta, \perp$ reduction,
- each β-normal form is also a normal form for the new $\perp$ rule.

The Berarducci tree of a term M is now the (tree of the) possibly infinite normal form of M for the β-$\perp$-reduction. In the present paper we will always identify terms with their trees.

Our starting point is the finite untyped lambda calculus (Barendregt, 1984). The set Λ of finite untyped λ-terms is given by the following inductive grammar:

$$M ::=_{\text{ind}} x \mid (\lambda x M) \mid (MM),$$

where x is a variable from some fixed countable set of variables $\mathcal{V}$. We follow the usual conventions on syntax. Terms and variables will respectively be written with (super- and subscripted) letters M, N and x, y, z. Terms of the form $(M_1 M_2)$ and $(\lambda x M)$ will respectively be called applications and abstractions. A context $C[\,]$ is a term with a hole in it, and $C[M]$ denotes the result of filling the hole by the term M, possibly by capturing some free variables of M. A term of the form $(\lambda x M)N$ is a β-redex.

We will silently take equivalence classes of terms modulo a change of bound variables and follow the variable naming convention (Barendregt, 1984)(2.1.13).

We will use the following abbreviations:

$$\lambda x_1 \ldots x_n.M =_{\text{def}} (\lambda x_1(\lambda x_2 \ldots (\lambda x_n M)\ldots))$$
$$MN_1 \ldots N_n =_{\text{def}} (\ldots(MN_1)\ldots N_n)$$
$$\mathbf{I} =_{\text{def}} \lambda x.x \quad \mathbf{S} =_{\text{def}} \lambda xyz.(xz)yz \quad \mathbf{K} =_{\text{def}} \lambda xy.x \quad \mathbf{B} =_{\text{def}} \lambda xyz.x(yz)$$
$$\Delta =_{\text{def}} \lambda x.xx \quad \Delta_\eta =_{\text{def}} \lambda x.x(\lambda y.xy) \quad \Delta_M =_{\text{def}} \lambda x.xxM$$
$$\Omega =_{\text{def}} \Delta\Delta \quad \Omega_M =_{\text{def}} \Delta_M \Delta_M$$
$$\mathbf{Y} =_{\text{def}} (\lambda xy.y(xxy))(\lambda xy.y(xxy))$$

The reduction relation $\to_\beta$ on Λ is the smallest binary relation that is closed under contexts[5] and contains the rule:

$$(\lambda x M)N \to M[N/x] \quad (\beta)$$

and $\to_\beta^*$ is its reflexive and transitive closure.

The structure of a λ-term can be described with help of the notions of β-zero, β-rootstable and β-rootactive term.

Definition 2 *(Berarducci, 1996) Let M be a λ-term in Λ. If M cannot β-reduce to an abstraction, then M is called a β-*zero *term.*

[5] A relation $\to_\rho$ is closed under contexts if $M \to_\rho N$ implies $C[M] \to_\rho C[N]$ for all contexts $C[\,]$.

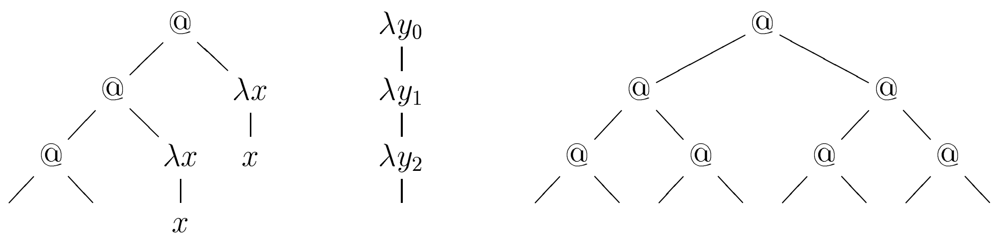


Fig. 1. Tree representation of the infinite β-normal forms of $\Omega_{\mathbf{I}}, \mathbf{YK}$ and $\mathbf{YY}$.

It is easy to verify that:

Lemma 3 *(Berarducci, 1996),(Kennaway et al., 1997) A λ-term β-reduces either to a variable, to an abstraction, to an application of the form MN where M is a β-zero term, or to a β-redex.*

Definition 4 *(Kennaway et al., 1997) Let M be a λ-term.*

(i) If M cannot β-reduce to a β-redex, then M is called β-rootstable *or a* β-rootstable form.
(ii) If for all N such that M can β-reduce to N, the term N can further be β-reduced to a β-redex, then M is called β-rootactive.

For example, Ω is a β-zero term and it is β-rootactive. The term **III** is an example of a term which is neither β-rootactive nor β-rootstable, because it can β-reduce to the β-rootstable term **I**.

Note that:

Lemma 5 *(Kennaway et al., 1997) A term can not β-reduce to a β-rootstable form if and only if it is β-rootactive.*

A λ-term has a *β-normal form* if it can β-reduce to a term that does not contain β-redexes anymore. Of course not all finite terms have a finite β-normal form. Some of these terms however seem to converge to an infinite β-normal form well beyond the scope of the finite lambda calculus. For example:

$$\Omega_{\mathbf{I}} \rightarrow_\beta \Omega_{\mathbf{I}}\mathbf{I} \rightarrow_\beta \Omega_{\mathbf{I}}\mathbf{II} \rightarrow_\beta \Omega_{\mathbf{I}}\mathbf{III} \rightarrow_\beta \ldots$$

$$\mathbf{YK} \rightarrow_\beta^* \mathbf{K}(\mathbf{YK}) \rightarrow_\beta \lambda y_0.\mathbf{YK} \rightarrow_\beta^* \lambda y_0 y_1.\mathbf{YK} \rightarrow_\beta^* \ldots$$

$$\mathbf{YY} \rightarrow_\beta^* \mathbf{Y}(\mathbf{YY}) \rightarrow_\beta^* \mathbf{YY}(\mathbf{YY}) \rightarrow_\beta^* \mathbf{YY}(\mathbf{YY})(\mathbf{YY}(\mathbf{YY})) \rightarrow_\beta^* \ldots$$

The infinite β-normal forms of these reductions can more clearly be repre-

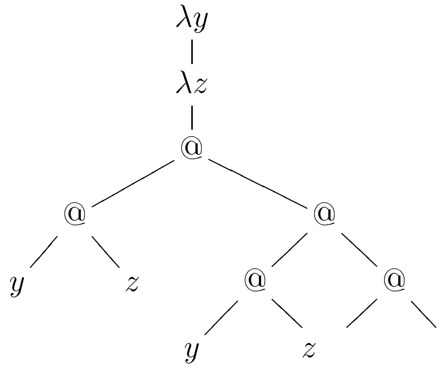


Fig. 2. Tree representation of the infinite β-normal forms of **BYS** and **BY**.

sented as planar trees instead of linear formulas, see Figure 1.[6]

For another example, one can calculate that **BYS** and **BY** β-reduce to the same infinite β-normal form $\lambda yz.yz(yz(yz(\ldots)))$ (see Figure 2). This shows that infinite reductions are an alternative to adding Scott's induction to the lambda calculus (Scott, 1975).[7]

2.2 Infinite Lambda Calculus

We will now recall the infinitary extension $\Lambda^{\infty}_{\perp}$ (Kennaway et al., 1997). It provides the proper context to introduce infinite λ-terms and converging reductions formally.

We first define the set $\Lambda_{\perp}$ of λ-terms extended with a constant $\perp$.

Definition 6 *The set $\Lambda_{\perp}$ of partial terms is defined by the* inductive *grammar:*

$$M :=_{ind} x \mid \perp \mid (\lambda x M) \mid (MM), \text{ where } x \in \mathcal{V}.$$

We give a coinductive definition of the set $\Lambda^{\infty}_{\perp}$ of infinite λ-terms which is equivalent to the one given in (Kennaway et al., 1997) and (Kennaway et al., 1999) as a metric completion.

[6] The one-one correspondence between terms and trees is given in Definition 13.
[7] Barendregt reformulated Scott's remark as an open problem: *Show that the equation* **BYS** = **BY** *cannot be proved in lambda calculus without induction.* These terms are proved to be equal in (Kennaway et al., 1997) without using induction, *provided* one replaces the finite zig and zags in the definition of β-conversion by strongly converging β-reductions. Scott's induction is then implicit in the definition of equality on the completion.

Definition 7 *The set of terms of the infinitary extension $\Lambda_\perp^\infty$ of the pure lambda calculus is defined by the* coinductive *grammar*[8] *:*

$$M ::=_{\mathrm{coind}} \perp \mid x \mid (\lambda x M) \mid (MM), \ \textit{where } x \in \mathcal{V}.$$

Notice that the grammar of $\Lambda_\perp$ differs from that of $\Lambda_\perp^\infty$ only for being inductive instead of coinductive.

We need an explicit definition of distance between two λ-terms in order to characterize $\Lambda_\perp^\infty$ as a metric completion, as it is defined in (Kennaway et al., 1997) (Kennaway et al., 1999), and to introduce the notion of converging reduction sequence.

Definition 8 *(i) Occurrences are finite words over the set $\{0, 1, 2\}$. Let $\langle\,\rangle$ denote the empty word.*

(ii) The subterm $M|_u$ of a term $M \in \Lambda_\perp^\infty$ at occurrence u is partially defined by induction on the length of u as usual:
- *(a) $M|\langle\,\rangle =_{\mathrm{def}} M$,*
- *(b) $(\lambda x M_0)|_{0u} =_{\mathrm{def}} M_0|_u$,*
- *(c) $(M_1 M_2)|_{1u} =_{\mathrm{def}} M_1|_u$,*
- *(d) $(M_1 M_2)|_{2u} =_{\mathrm{def}} M_2|_u$.*

Note that the term $M|_u$ may not exist. If it exists, then u is an occurrence *of M.*

(iii) The depth *of a subterm N at occurrence u of $M \in \Lambda_\perp^\infty$ is the length of the occurrence u.*

(iv) The distance $d(M, N)$ of two terms $M, N \in \Lambda_\perp^\infty$ is 0 if M and N are identical and it is 2^{-k} if k is the length of the shortest occurrence u such that $M|_u$ and $N|_u$ exist and differ.

With this distance $\Lambda_\perp^\infty$ becomes a metric space: it is easy to verify that $\Lambda_\perp^\infty$ is the metric completion of the set $\Lambda_\perp$.

We skip the details of extending substitution to infinite terms and refer to Definition 2 of (Kennaway et al., 1997).

We extend some concepts related to β-reduction from Λ to $\Lambda_\perp^\infty$.

Definition 9 *(i) The reduction relation $\rightarrow_\beta$ on $\Lambda_\perp^\infty$ is the smallest binary relation that is closed under contexts and contains the rule:*

$$(\lambda x M)N \rightarrow M[N/x] \quad (\beta)$$

[8] In fact $\Lambda_\perp^\infty$ is the final coalgebra of the polynomial endofunctor $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ defined by $F(X) = 1 + \mathcal{V} + \mathcal{V} \times X + X \times X$, where $\mathcal{V}$ is the set of variables. See (Barr, 1993) for the categorical background.

(ii) *If $M \in \Lambda^\infty_\perp$ cannot β-reduce to an abstraction, then M is called a β-*zero term.

(iii) *A term $M \in \Lambda^\infty_\perp$ is called β-*rootstable *if $M[\Omega/\perp]$ cannot β-reduce to a β-redex.*

(iv) *A term $M \in \Lambda^\infty_\perp$ is called β-*rootactive *if for all $N \in \Lambda^\infty_\perp$ such that $M[\Omega/\perp]$ can β-reduce to N, the term N can further be β-reduced to a β-redex.*

(v) *$R^\infty_\perp$ is the set of β-rootactive terms in $\Lambda^\infty_\perp$.*

(vi) *The reduction relation $\rightarrow_{\beta\perp}$ on $\Lambda^\infty_\perp$ is the smallest binary relation that is closed under contexts and contains the two rules:*

$$(\lambda x M)N \rightarrow M[N/x] \quad (\beta) \qquad\qquad \frac{M \neq \perp \text{ and } \beta\text{-rootactive}}{M \rightarrow \perp}\,(\perp)$$

Note that $\perp$ is a β-rootactive term, since $\perp[\Omega/\perp] = \Omega$ and Ω is β-rootactive.

Definition 10 *(i) An infinite reduction $M_0 \rightarrow_{\beta\perp} M_1 \rightarrow_{\beta\perp} M_2 \rightarrow_{\beta\perp} \ldots$ is Cauchy converging with limit M_ω (notation $\lim_{n\rightarrow\omega} M_n = M_\omega$) if $\forall\epsilon > 0.\exists n.\forall k \geq n.d(M_k, M_\omega) < \epsilon$.*

(ii) *An infinite reduction $M_0 \rightarrow_{\beta\perp} M_1 \rightarrow_{\beta\perp} M_2 \rightarrow_{\beta\perp} \ldots$ is strongly converging with limit M_ω if $\lim_{n\rightarrow\omega} M_n = M_\omega$ and $\lim_{n\rightarrow\omega} d_n = \omega$, that is, $\forall n.\exists m.\forall k \geq m.d_k > n$, where d_k denotes the depth of the redex at occurrence u in M_k reduced in the reduction step $M_k \rightarrow_{\beta\perp} M_{k+1}$.*

(iii) *We say that a term M has a possibly infinite β-$\perp$-reduction to N (notation $M \twoheadrightarrow_{\beta\perp} N$) if either there is a finite β-$\perp$-reduction $M \rightarrow^*_{\beta\perp} N$ or there is a strong converging β-$\perp$-reduction starting from M with limit N.*

It is well known that without rule $\perp$ strongly converging reductions jeopardize the confluence property for β-reduction. Unlike finite reductions, Cauchy converging and even strongly converging β-reductions are not confluent (Berarducci, 1996), (Kennaway et al., 1995), (Kennaway et al., 1997). The finite term $\mathbf{Y}(\lambda z.\mathbf{K}(\mathbf{K}zy)x)$ can converge in an infinite β-reduction to the infinite term $\mathbf{K}(\mathbf{K}(\ldots x)x)$ not containing y. It can also converge to the infinite term $\mathbf{K}(\mathbf{K}(\ldots y)y)$ that does not contain x. Both terms can not be joined; they can only β-reduce to themselves. A simpler example (Berarducci, 1996) is the term $(\lambda x.\mathbf{I}(xx))(\lambda x.\mathbf{I}(xx))$ which reduces to both Ω and to $\mathbf{I}(\mathbf{I}\ldots))$; also these two terms cannot be joined.

Strongly converging reductions are Cauchy convergent, but not conversely. For example, $\Omega \rightarrow_\beta \Omega \rightarrow_\beta \ldots$ is weakly convergent but not strongly convergent, as the depth of the reduced redexes is always zero.

We recall here the crucial properties of $\twoheadrightarrow_{\beta\perp}$ reduction which will be useful in the following and are proved in (Berarducci, 1996), (Kennaway et al., 1997), (Kennaway and de Vries, 2000), and we refer the reader to those papers to know more on this subject. In particular the interested reader will find there that $\twoheadrightarrow_{\beta\bot}$ reduction has been defined for sequences of transfinite ordinal length. However these can be compressed into similarly converging reductions of at most ω length with same initial and final terms. The compression lemma of $\twoheadrightarrow_{\beta\bot}$ for terms in $\Lambda^\infty_\bot$ easily generalizes to $\twoheadrightarrow_{\beta\bot}$-reductions of terms in the extensions considered later in this paper.

Theorem 11 *(Berarducci, 1996) (Kennaway et al., 1997) (Kennaway and de Vries, 2000)*

(i) If a term in $\Lambda^\infty_\bot$ has a β-rootstable form then such a form can be computed in finitely many steps.
(ii) The reduction $\twoheadrightarrow_{\beta\bot}$ is confluent.
(iii) Every term in $\Lambda^\infty_\bot$ has a unique β-$\bot$-normal form.

2.3 Berarducci Trees as Normal Forms in $\Lambda^\infty_\bot$

In this section we give the central definition of this paper, i.e. the definition of Berarducci tree. Since the notion of Berarducci tree will be given as a corecursive function, the codomain of this function has to be given by coinduction. Hence we first define the codomain of this function, i.e. the set of trees.

Definition 12 *The set of* trees *is defined by the coinductive grammar:*

$$T ::=_{\text{coind}} \bot \mid x \mid \begin{array}{c}\lambda x\\ \mid \\ T\end{array} \mid \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ T & & T\end{array}, \text{ where } x \in \mathcal{V}.$$

It is not difficult to show that this notion of tree is a particular case of the notion of Σ-labelled tree defined in (Barendregt, 1984) (Definition 10.1.1) as a partial map from the set of sequence numbers to Σ, where $\Sigma = \{x, \lambda x \mid x \in \mathcal{V}\} \cup \{\bot, @\}$. In our terminology, a tree is a partial map from the set of occurrences (see Definition 8(i)) to Σ.

Definition 13 gives a natural one-one correspondence between trees and terms of $\Lambda^\infty_\bot$. So in the following we will freely identify trees and terms of $\Lambda^\infty_\bot$.

Definition 13 *The* tree $\mathcal{T}(M)$ *of the term $M \in \Lambda^\infty_\bot$ is defined by corecursion:*
$\mathcal{T}(\bot) = \bot$;
$\mathcal{T}(x) = x$;

$$\mathcal{T}(\lambda x.N) = \begin{array}{c} \lambda x \\ | \\ \mathcal{T}(N) \end{array} \quad ;$$

$$\mathcal{T}(M_1 M_2) = \begin{array}{ccc} & @ & \\ \swarrow & & \searrow \\ \mathcal{T}(M_1) & & \mathcal{T}(M_2) \end{array} \quad .$$

We can now give the definition of Berarducci tree in a graphically pleasing tree format in the spirit of Barendregt's definition of Böhm tree (Barendregt, 1984).

Definition 14 *(Berarducci, 1996) The* Berarducci tree $\mathcal{B}e\mathcal{T}(M)$ *of a term* $M \in \Lambda^{\infty}_{\perp}$ *can be constructed via the following corecursive procedure:*

(i) if $M \to^*_\beta x$*, then* $\mathcal{B}e\mathcal{T}(M) = x$*;*

(ii) if $M \to^*_\beta \lambda x.N$*, then* $\mathcal{B}e\mathcal{T}(M) = \begin{array}{c} \lambda x \\ | \\ \mathcal{B}e\mathcal{T}(N) \end{array}$ *;*

(iii) if $M \to^*_\beta M_1 M_2$*, where* M_1 *is a* β*-zero term, then*

$$\mathcal{B}e\mathcal{T}(M) = \begin{array}{ccc} & @ & \\ \swarrow & & \searrow \\ \mathcal{B}e\mathcal{T}(M_1) & & \mathcal{B}e\mathcal{T}(M_2) \end{array} \quad ;$$

(iv) otherwise, (exactly when M *is* β*-rootactive)* $\mathcal{B}e\mathcal{T}(M) = \perp$*.*

The infinitary lambda calculus with $\perp$-rule $\Lambda^{\infty}_{\perp}$ is an extension of the lambda calculus that has been so designed that in this extension the Berarducci tree of a term is nothing else but its unique (possibly infinite) β-$\perp$-normal form that can be found by a possibly infinite reduction.

Theorem 15 *(Berarducci, 1996)(Kennaway et al., 1997) The* Berarducci tree $\mathcal{B}e\mathcal{T}(M)$ *of a term* $M \in \Lambda^{\infty}_{\perp}$ *is the unique* β*-*$\perp$*-normal form* N *such that* $M \twoheadrightarrow_{\beta\perp} N$*.*

The following result connects Berarducci trees with contexts. It plays a crucial role in this paper:

Theorem 16 *(de Vries, 1997) For all terms* M *and contexts* $C[\ ]$ *in* $\Lambda^{\infty}_{\perp}$ *it holds that* $\mathcal{B}e\mathcal{T}(C[M]) = \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(M)])$.

The proof is simple: just recognize that the left hand side and the right hand side of the equation represent two ways of reducing to the unique β-$\perp$-normal form of $C[M]$.

Böhm Trees (Barendregt, 1984) and Lévy-Longo trees (Lévy, 1976), (Longo, 1983), (de Vries, 1997) can be seen as normal forms in similar extensions as $\Lambda^\infty_\perp$. The extensions use the same syntax and β-rule but have both different and more $\perp$-rules. In all cases the basic idea is that terms "without computational value" will be replaced by $\perp$.

In (Kennaway et al., 1997), (Kennaway et al., 1999) candidate sets of terms with no computational value that lead to calculi with infinite confluence properties have been systematically investigated for lambda calculus. Three such sets resulted: the set of terms without *β-head normal form*[9] (Barendregt, 1984), (Wadsworth, 1976), the set of terms without *weak β-head normal form*[10] (Abramsky and Ong, 1993) and the set of *β-rootactive terms.*

The reduction relation $\to_{\text{Böhm}}$ on $\Lambda^\infty_\perp$ is the smallest binary relation that is closed under contexts and contains the four rules:

$$(\lambda x M)N \to M[N/x] \quad (\beta)$$

$$M \to \perp, \text{ provided } M \neq \perp \text{ and has no } \beta\text{-head normal form} \quad (\perp)$$

$$\lambda x.\perp \to \perp \quad (\perp_\lambda)$$

$$\perp M \to \perp \quad (\perp\text{app})$$

The normal forms of λ-terms with these rules are better know as *Böhm trees.*

Definition 17 *(Barendregt, 1984) The* Böhm tree $\mathcal{B}\ddot{o}\mathcal{T}(M)$ *of a λ-term M can be constructed via the following corecursive procedure:*

(i) if $M \to^*_\beta \lambda x_1 \ldots x_n.yM_1 \ldots M_k$, then

$$\mathcal{B}\ddot{o}\mathcal{T}(M) = \begin{array}{ccc} & \lambda x_1 \ldots x_n.y & \\ \swarrow & & \searrow \\ \mathcal{B}\ddot{o}\mathcal{T}(M_1) & \ldots & \mathcal{B}\ddot{o}\mathcal{T}(M_k) \end{array};$$

(ii) otherwise, when M has no β-head normal form $\mathcal{B}\ddot{o}\mathcal{T}(M) = \perp$.

The reduction reduction relation $\to_{\text{LeLo}}$ on $\Lambda^\infty_\perp$ is the smallest binary relation

[9] A λ-term has a β-head normal form when it β-reduces to a term of the form $\lambda x_1 \ldots x_n.yM_1 \ldots M_k$.

[10] A λ-term has a weak β-head normal form when it β-reduces to a term of the form $\lambda x.M$ or $yM_1 \ldots M_k$.

that is closed under contexts and contains the three rules:

$$(\lambda x M)N \to M[N/x] \quad (\beta)$$

$$M \to \bot, \text{ provided } M \neq \bot \text{ and has no weak } \beta\text{-head normal form} \quad (\bot)$$

$$\bot M \to \bot \quad (\bot\mathrm{app})$$

The normal forms of λ-terms with these rules are better known as *Lévy-Longo trees.*

Definition 18 *(Lévy, 1976), (Longo, 1983) The* Lévy-Longo tree $\mathcal{LLT}(M)$ *of a λ-term M can be constructed via the following corecursive procedure:*

(i) if $M \to_\beta^* xM_1 \ldots M_k$*, then* $\mathcal{LLT}(M) = \begin{array}{c} x \\ \diagup \quad \diagdown \\ \mathcal{LLT}(M_1) \quad \ldots \quad \mathcal{LLT}(M_k) \end{array}$;

(ii) if $M \to_\beta^* \lambda x.N$*, then* $\mathcal{LLT}(M) = \begin{array}{c} \lambda x \\ | \\ \mathcal{LLT}(N) \end{array}$;

(iii) otherwise, when M has no weak β-head normal form $\mathcal{LLT}(M) = \bot$.

Comparing the tree formats we find that the Lévy-Longo tree of a term reveals at least the same computational content of a term as its Böhm tree does. The Böhm tree of **YK** is just $\bot$, as **YK** does not have a head normal form. In contrast, the Lévy-Longo tree of **YK** is the infinite term in the centre of Figure 1. Lévy-Longo trees don't include all infinite normal forms of the finite lambda calculus: the Lévy-Longo tree of **YY** is $\bot$ and not the term depicted in Figure 1. Berarducci trees are very nice from a theoretical point of view in that they provide the maximal [11] "computational" value of a term. From a practical point of view they seem to be less useful than Böhm trees and Lévy-Longo trees, being it undecidable whether a term is a β-zero term or not.

3 The Extended Calculus $\Lambda^\infty_{\bot\mathsf{OA}}$

The notion of Berarducci tree gives an *equivalence relation*: two terms in $\Lambda^\infty_\bot$ are equivalent if and only if they have the same Berarducci tree (modulo α-conversion, as defined in (Kennaway et al., 1997)). A completely different way of comparing terms in $\Lambda^\infty_\bot$ is *observational equivalence* (Morris Jr., 1968), in

[11] The set of β-rootactive terms is the smallest set of terms in Λ which can be mapped into $\bot$, such that the corresponding $\Lambda^\infty_\bot$ has the unique β-$\bot$-normal form property (Kennaway et al., 1997).

which we say that M is equivalent to N if:

$$\forall C[\,] \in \Lambda\ (C[M] \text{ is } \beta\text{-rootactive } \Leftrightarrow C[N] \text{ is } \beta\text{-rootactive}).$$

Here we put M and N in various contexts and observe whether the behavior of M and N in those contexts is the same, that is whether $C[M]$ and $C[N]$ are both β-rootactive terms. Berarducci tree equality implies observational equivalence (de Vries, 1997):

Theorem 19 *For all $M, N \in \Lambda_\perp^\infty$, $\mathcal{B}e\mathcal{T}(M) = \mathcal{B}e\mathcal{T}(N)$ implies $\forall C[\,] \in \Lambda$ $C[M] \in R_\perp^\infty \Leftrightarrow C[N] \in R_\perp^\infty$.*

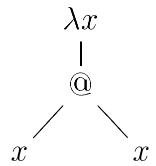
PROOF. This can be easily seen with help of Theorem 16 which for any term M and context $C[\,]$ says:

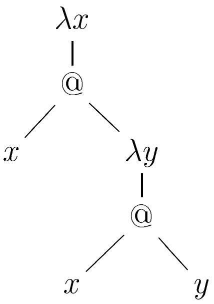
$$\mathcal{B}e\mathcal{T}(C[M]) = \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(M)]).$$

If M, N are two terms with the same Berarducci tree, then we find that

$$\begin{aligned} C[M] \in R_\perp^\infty &\Leftrightarrow \mathcal{B}e\mathcal{T}(C[M]) = \perp \Leftrightarrow \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(M)]) = \perp \Leftrightarrow \\ &\Leftrightarrow \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(N)]) = \perp \Leftrightarrow \mathcal{B}e\mathcal{T}(C[N]) = \perp \Leftrightarrow C[N] \in R_\perp^\infty. \end{aligned}$$

In a similar way we can prove that Böhm and Lévy-Longo tree equality implies observational equivalence.

The converse of Theorem 19 is not true: observational equivalence does not imply Berarducci tree equivalence. We show two examples of different nature. The first one shows that η-convertible terms cannot be discriminated. Consider the η-convertible terms Δ and Δ_η whose Berarducci trees are 

and  . Since both terms have no free variables, it is enough to show that for all $M \in \Lambda$, $M[x := \Delta] \in R_\perp^\infty \Leftrightarrow M[x := \Delta_\eta] \in R_\perp^\infty$. This is proved by case analysis. We know that $M \to_\beta^* N$ where N is either a variable, an abstraction, an application whose operator is a β-zero term or a β-rootactive term. The interesting case is when N is an application of the form $xP_1 \ldots P_n$. If P_1 is a β-zero term then $\Delta P_1 \ldots P_n \to_\beta P_1 P_1 \ldots P_n$, $\Delta_\eta P_1 \ldots P_n \to_\beta P_1(\lambda y.P_1 y) \ldots P_n$, and both terms are β-rootstable. If P_1 β-reduces to an abstraction then $\Delta P_1 \ldots P_n$ and $\Delta_\eta P_1 \ldots P_n$ are β-convertible.

The second example shows that even if we consider Berarducci trees modulo η-expansions, the converse of Theorem 19 is not true. Consider the terms $\Omega\Omega$ and $\Omega\Omega\Omega$ whose Berarducci trees are $\begin{array}{ccc} & @ & \\ / & & \backslash \\ \perp & & \perp \end{array}$ and $\begin{array}{ccccc} & & & @ & \\ & & / & & \backslash \\ & @ & & & \perp \\ / & & \backslash & & \\ \perp & & \perp & & \end{array}$. It is enough to prove that for all $M \in \Lambda$, $M[x := \Omega\Omega] \in R^\infty_\perp \Leftrightarrow M[x := \Omega\Omega\Omega] \in R^\infty_\perp$. The proof proceeds by case analysis similarly to the previous example.

In order to obtain the converse of Theorem 19, we will extend the lambda calculus with two new symbols and four new rules.

3.1 Syntax

Associated with the pure lambda calculus Λ and its extensions $\Lambda_\perp, \Lambda^\infty_\perp$ we define the extensions $\Lambda_{\mathsf{OA}}, \Lambda_{\perp\mathsf{OA}}$ and $\Lambda^\infty_{\perp\mathsf{OA}}$ with the constants $\perp, \mathsf{O}$ and A. First we introduce the syntax of these sets and then the reduction rules.

Definition 20 *(i) The extension Λ_{OA} of Λ with the constants O, A is defined by the* inductive *grammar:*

$$V :=_{ind} P \mid (\mathsf{O}V) \mid (\mathsf{A}V) \mid (\lambda x V) \mid (VP), \ \textit{where } P \in \Lambda \ \textit{ is closed}$$
$$X :=_{ind} M \mid V, \quad \textit{where} \ \ M \in \Lambda.$$

(ii) The extension $\Lambda_{\perp\mathsf{OA}}$ of Λ_{OA} with partial terms is defined by the inductive *grammar:*

$$V :=_{ind} P \mid (\mathsf{O}V) \mid (\mathsf{A}V) \mid (\lambda x V) \mid (VP), \ \textit{where } P \in \Lambda_\perp \ \textit{ is closed}$$
$$X :=_{ind} M \mid V, \quad \textit{where} \ \ M \in \Lambda_\perp.$$

(iii) The infinitary extension $\Lambda^\infty_{\perp\mathsf{OA}}$ of Λ_{OA} is defined by the inductive *grammar:*

$$V :=_{ind} P \mid (\mathsf{O}V) \mid (\mathsf{A}V) \mid (\lambda x V) \mid (VP), \ \textit{where } P \in \Lambda^\infty_\perp \ \textit{ is closed}$$
$$X :=_{ind} M \mid V, \quad \textit{where} \ \ M \in \Lambda^\infty_\perp.$$

These extensions with two new constants O and A are rather minimal since the syntax definition implies the following conditions:

- a term can contain only finitely many occurrences of O and A,
- O and A themselves are not terms,
- O and A can only be applied to closed terms and
- O and A can occur in the argument of an application only if the operator is O or A.

For example, $\lambda x.\mathsf{O}(\mathsf{A}\mathbf{I}) \in \Lambda_{\perp\mathsf{OA}}$ but $\mathbf{I}(\mathsf{A}\mathbf{I}) \notin \Lambda_{\perp\mathsf{OA}}$.

3.2 Rewrite Rules

We introduce now the reduction relations of the various calculi in a concise form. We will use some standard notational conventions. Let Λ' be some extension of the set Λ.

Definition 21 *Let $\to_1$ and $\to_2$ be reduction relations on Λ'.*

(i) The reduction relation $\to_{12}$ is defined as the union of the reduction $\to_1$ with $\to_2$.
(ii) The reduction relation $\to_1^=$ is the reflexive closure of $\to_1$.
(iii) The reduction relation $\to_1^*$ is the reflexive and transitive closure of $\to_1$.

In what follows, we need the notions of β-zero, β-rootstable and β-rootactive term given in Definition 9. We now introduce the notion of OA-uniform term that will be used in Definition 24 . The idea behind this notion is that the constants O and A applied to certain terms called OA-uniform will behave like "constant functions".

Definition 22 *Let $\Lambda_\perp^\infty \supseteq \Lambda' \supseteq \Lambda$. We say that a term in Λ' is* OA-uniform *if it is either an abstraction or a β-rootactive term.*

Proposition 23 *Let $\Lambda_\perp^\infty \supseteq \Lambda' \supseteq \Lambda$ and M a closed term in Λ'. Then $M \to_\beta^* N$ where N is either a OA-uniform term or a β-rootstable application.*

PROOF. A closed term either β-reduces to an abstraction, to a β-rootactive term, or to PQ where P is a closed β-zero term.

Now we introduce the reduction rules for the constants O and A that will allow us to discriminate terms using the Böhm-out technique. If a closed term is a β-rootstable application then the constant O selects the operator of the application and A selects the argument. On the other hand, if the closed term is OA-uniform the constants O and A behave as "constants functions".

Definition 24 *Let $\Lambda' \supseteq \Lambda_{\mathsf{OA}}$ and M, N be closed terms of Λ' that do not contain O or A.*

(i) We define the reduction relations $\to_{\mathsf{OA}}$ on Λ' as the smallest binary relation that is closed under contexts and contains the following rules:

$$\frac{M \text{ is } \beta\textit{-zero}}{\mathsf{O}(MN) \to M}\ (\mathsf{O}\textit{-selection}) \qquad \frac{M \text{ is } \beta\textit{-zero}}{\mathsf{A}(MN) \to N}\ (\mathsf{A}\textit{-selection})$$

$$\frac{M \text{ is } \mathsf{OA}\textit{-uniform}}{\mathsf{O}M \to \mathbf{I}}\ (\mathsf{O}\textit{-constant}) \qquad \frac{M \text{ is } \mathsf{OA}\textit{-uniform}}{\mathsf{A}M \to \mathbf{I}}\ (\mathsf{A}\textit{-constant})$$

(ii) An OA-redex *is a term in Λ' of the form $\mathsf{O}M$ or $\mathsf{A}M$ where M is any closed term in Λ' that does not contain O or A.*

In (Dezani-Ciancaglini et al., 2000), the constants O and A only performed respectively the selection of the operator and of the argument of an application by rules O-selection and A-selection. By adding the rules O-constant and A-constant, terms having different Berarducci tree in (Dezani-Ciancaglini et al., 2000) are equated by reduction. For example, the terms $\mathsf{O}(\mathsf{A}\Omega)$ and $\mathsf{O}(\mathsf{O}\Omega)$ were different normal forms in (Dezani-Ciancaglini et al., 2000) and now they both OA-reduce to $\mathbf{I}$.

Example 25 *Let $\Lambda' \supseteq \Lambda_{\perp}$. The fact that $\perp$ is a closed β-zero term implies that:*

(i) $\mathsf{O}(\perp M_1 \dots M_n N) \to_{\mathsf{OA}} \perp M_1 \dots M_n$
(ii) $\mathsf{A}(\perp M_1 \dots M_n N) \to_{\mathsf{OA}} N$

for all $M_1, \dots, M_n, N$ closed λ-terms of Λ' ($n \geq 0$).

It is easy to show that the reduction $\to_{\beta\mathsf{OA}}$ eliminates all occurrences of the constants O and A.

Lemma 26 *Let $X \in \Lambda^{\infty}_{\perp\mathsf{OA}}$. Then there is $M \in \Lambda^{\infty}_{\perp}$ such that $X \to^{*}_{\beta\mathsf{OA}} M$.*

PROOF. The proof by induction on the definition of $\Lambda^{\infty}_{\perp\mathsf{OA}}$ using Proposition 23 is easy.

We generalize the notions of β-zero, β-rootstable and β-rootactive term given in Definition 9. We say that a $\beta\mathsf{OA}$-redex is either a β-redex or an OA-redex.

Definition 27 *Let $\Lambda' \supseteq \Lambda_{\mathsf{OA}}$ and ρ be β or $\beta\mathsf{OA}$.*

(i) *We say that a term in Λ' is ρ-*zero *if it cannot ρ-reduce to an abstraction.*
(ii) *We say that a term $X \in \Lambda'$ is ρ-*rootstable *if $X[\Omega/\bot]$ cannot ρ-reduce to a ρ-redex.*
(iii) *We say that a term $X \in \Lambda'$ is ρ-*rootactive *if all the reducts of $X[\Omega/\bot]$ can ρ-reduce to ρ-redexes.*

Notice that a term is ρ-rootactive if and only if it can not ρ-reduce to a ρ-rootstable term.

A short notation for the set of terms in $\Lambda_{\mathsf{OA}}, \Lambda^{\infty}_{\bot\mathsf{OA}}$ which are $\beta\mathsf{OA}$-rootactive will be handy.

Definition 28 *(i)* *R_{OA} is the set of of terms in Λ_{OA} which are $\beta\mathsf{OA}$-rootactive.*
(ii) *$R^{\infty}_{\bot\mathsf{OA}}$ is the set of of terms in $\Lambda^{\infty}_{\bot\mathsf{OA}}$ which are $\beta\mathsf{OA}$-rootactive.*

We can characterize the set of rootstable terms for the extended set $\Lambda^{\infty}_{\bot\mathsf{OA}}$ and the reduction $\to_{\beta\mathsf{OA}}$.

Proposition 29 *A term X in $\Lambda^{\infty}_{\bot\mathsf{OA}}$ is $\beta\mathsf{OA}$-rootstable if and only if it has one of the following shapes:*

- *a variable,*
- *an abstraction,*
- *an application of the form MN with M a β-zero term.*

We can also say that $\beta\mathsf{OA}$-rootactivity and β-rootactivity coincide in the following sense:

Lemma 30 *(i)* *If $X \in \Lambda^{\infty}_{\bot\mathsf{OA}}$ is $\beta\mathsf{OA}$-rootactive then there exists a term $M \in \Lambda^{\infty}_{\bot}$ such that $X \to^{*}_{\beta\mathsf{OA}} M$ and M is β-rootactive.*
(ii) *If $X \in \Lambda^{\infty}_{\bot\mathsf{OA}}$ is β-rootactive then $X \in \Lambda^{\infty}_{\bot}$.*

PROOF.

(i) It follows from Lemma 26.
(ii) If $X \notin \Lambda^{\infty}_{\bot}$ then $X = (\lambda x_1 \ldots x_n.Y)M_1 \ldots M_k$ and either O or A occur in the head position of Y. Hence either X β-reduces to an abstraction or to an application whose head is O or A. In both cases, X is β-rootstable.

The last rule we introduce allows us to equate all rootactive terms.

Definition 31 *Let $\Lambda' \supseteq \Lambda_{\bot}$. We define the reduction relation $\to_{\bot}$ as the smallest binary relation on Λ' that is closed under contexts and contains the* *rule:*

$$\frac{X \neq \bot \quad \textit{and } X \textit{ is } \ \beta\textit{-rootactive}}{X \to \bot}\,(\bot)$$

where X *ranges over* Λ'.

We will now first consider the combinations $(\to_{\beta\mathsf{OA}}, \Lambda_{\mathsf{OA}})$ and $(\to_{\beta\bot\mathsf{OA}}, \Lambda_{\bot\mathsf{OA}})$, and later $(\to_{\beta\bot\mathsf{OA}}, \Lambda^{\infty}_{\bot\mathsf{OA}})$.

3.3 Confluence of Finite Reductions in Λ_{OA} *and* $\Lambda_{\bot\mathsf{OA}}$

We use the Hindley-Rosen Lemma (Proposition 3.3.5 of (Barendregt, 1984)) to prove that the reduction relations $\to^*_{\beta\mathsf{OA}}$ on Λ_{OA} and $\to^*_{\beta\bot\mathsf{OA}}$ on $\Lambda_{\bot\mathsf{OA}}$ are confluent. We need a few auxiliary lemmas.

Proposition 32 *There is at most one* OA*-redex in a term belonging to* $\Lambda_{\bot\mathsf{OA}}$. *Hence* $\to_{\mathsf{OA}}$ *is trivially confluent.*

PROOF. By induction on the definition of $\Lambda_{\bot\mathsf{OA}}$.

Lemma 33 *(i) The relation* $\to^*_{\beta}$ *is confluent in* Λ_{OA}.
(ii) The relation $\to^*_{\beta\bot}$ *is confluent in* $\Lambda_{\bot\mathsf{OA}}$.

PROOF. Because the O and A symbols are not reduced they can be thought of as fresh free variables. More precisely: $X \to_\beta Y$ if and only if $X[\mathsf{O}/x, \mathsf{A}/y] \to_\beta Y[\mathsf{O}/x, \mathsf{A}/y]$ for all $X \in \Lambda_{\mathsf{OA}}$ and $X \to_{\beta\bot} Y$ if and only if $X[\mathsf{O}/x, \mathsf{A}/y] \to_{\beta\bot} Y[\mathsf{O}/x, \mathsf{A}/y]$ for all $X \in \Lambda_{\bot\mathsf{OA}}$. Hence part (i) follows from the confluence property for $\to_\beta$ in Λ (see (Barendregt, 1984), Theorem 3.28) and part (ii) follows from the confluence property for $\to_{\beta\bot}$ in $\Lambda^{\infty}_{\bot}$ (see (Kennaway et al., 1997), (Kennaway et al., 1999)).

Lemma 34 *(i) The relation* $\to^*_{\mathsf{OA}}$ *commutes with the relation* $\to^*_{\beta}$ *in* Λ_{OA}*:*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "\mathsf{OA}"'] & Y_1 \arrow[d, dotted, "\mathsf{OA}"] \\ Y_2 \arrow[r, dotted, "\beta"'] & Z \end{tikzcd}$$

(ii) The relation $\to^*_{\mathsf{OA}}$ *commutes with the relation* $\to^*_{\beta\bot}$ *in* $\Lambda_{\bot\mathsf{OA}}$*:*

$$
\begin{tikzcd}
X \arrow[r, "\beta\bot"] \arrow[d, "\mathsf{OA}"'] & Y_1 \arrow[d, dotted, "\mathsf{OA}"] \\
Y_2 \arrow[r, dotted, "\beta\bot"', "=" near end] & Z
\end{tikzcd}
$$

PROOF. We give the proof for $\to_{\beta\bot\mathsf{OA}}$. The proof for $\to_{\beta\mathsf{OA}}$ is similar, just drop all references to $\to_{\bot}$. Suppose that $C[\mathsf{A}M] \to_{\mathsf{OA}} C[N]$ and $C[\mathsf{A}M] \to_{\beta\bot} X$. We distinguish four cases depending on the shape of M and whether the $\to_{\beta\bot}$ reduction reduces a subterm in $C[\ ]$ or in M.

– A $\to_{\beta\bot}$ reduction step in $C[\ ]$ can cause substitutions of variables inside $C[\ ]$. Since M does not contain free variables, it remains unchanged. Hence the resulting term will be of the form $C'[\mathsf{A}M]$.

$$
\begin{tikzcd}
C[\mathsf{A}M] \arrow[r, "\beta\bot"] \arrow[d, "\mathsf{OA}"'] & C'[\mathsf{A}M] \arrow[d, "\mathsf{OA}"] \\
C[N] \arrow[r, "\beta\bot"'] & C'[N]
\end{tikzcd}
$$

– If M is OA-uniform, a $\to_{\beta\bot}$ reduction step in M does not affect the redex $\mathsf{A}M$, because OA-uniform terms are closed under $\to_{\beta\bot}$. This gives us the diagram:

$$
\begin{tikzcd}
C[\mathsf{A}M] \arrow[r, "\beta\bot"] \arrow[d, "\mathsf{OA}"'] & C[\mathsf{A}M'] \arrow[d, "\mathsf{OA}"] \\
C[\mathbf{I}] \arrow[r, equal] & C[\mathbf{I}]
\end{tikzcd}
$$

– If $M \equiv PN$ where P is a β-zero term, a $\to_{\beta\bot}$ reduction step in P does not affect the redex $\mathsf{A}(PN)$, because β-zero terms are closed under $\to_{\beta\bot}$. This gives us the diagram:

$$
\begin{tikzcd}
C[\mathsf{A}(PN)] \arrow[r, "\beta\bot"] \arrow[d, "\mathsf{OA}"'] & C[\mathsf{A}(P'N)] \arrow[d, "\mathsf{OA}"] \\
C[N] \arrow[r, equal] & C[N]
\end{tikzcd}
$$

– Finally if $M \equiv PN$ where P is a β-zero term, a $\to_{\beta\bot}$-reduction step in N commutes trivially:

$$
\begin{tikzcd}
C[\mathsf{A}(PN)] \arrow[r, "\beta\bot"] \arrow[d, "\mathsf{OA}"'] & C[\mathsf{A}(PN')] \arrow[d, "\mathsf{OA}"] \\
C[N] \arrow[r, "\beta\bot"'] & C[N']
\end{tikzcd}
$$

The proof for the cases involving O is similar.

Theorem 35 *(i) The relation $\rightarrow^*_{\beta\mathsf{OA}}$ is confluent in Λ_{OA}.*
(ii) The relation $\rightarrow^*_{\beta\perp\mathsf{OA}}$ is confluent in $\Lambda_{\perp\mathsf{OA}}$.

PROOF. The Hindley-Rosen Lemma (Barendregt, 1984) states that if we know that two reduction relations $\rightarrow^*_1$ and $\rightarrow^*_2$ both are confluent, and that $\rightarrow^*_1$ commutes with $\rightarrow^*_2$, then $\rightarrow^*_{12}$ is confluent. Lemmas 32, 33 and 34 imply these conditions both for $\rightarrow^*_\beta$ and $\rightarrow^*_{\mathsf{OA}}$, and for $\rightarrow^*_{\beta\perp}$ and $\rightarrow^*_{\mathsf{OA}}$.

Remark 36 The extended calculus Λ_{OA} and the new reduction rules were chosen carefully in order to get confluent reduction relations. If the O-selection rule could be applied to open β-zero terms, then $(\lambda x.\mathsf{O}(x\mathbf{I}))\mathbf{K}$ would reduce to both $\mathbf{K}$ and $\mathbf{I}$. If the O-constant rule could be applied to open β-zero terms, then $(\lambda x.\mathsf{O}(x\mathbf{I}))\Omega$ would reduce to both $\mathbf{I}$ and Ω. In both cases, we would loose confluence.

4 Tree Equality implies Observational Equivalence

The goal of this section is to prove along similar lines as for $\Lambda^\infty_\perp$ (Theorem 19) that Berarducci tree equality in $\Lambda^\infty_{\perp\mathsf{OA}}$ implies observational equivalence in $\Lambda^\infty_{\perp\mathsf{OA}}$.

Our first step is to define $\twoheadrightarrow_{\beta\perp\mathsf{OA}}$-reductions for $\Lambda^\infty_{\perp\mathsf{OA}}$ and show that these reductions are infinitary confluent. Because terms in $\Lambda^\infty_{\perp\mathsf{OA}}$ contain at most a finite number of symbols O and A we can base the proof on the infinitary confluence of $\Lambda^\infty_\perp$ via a few straightforward lemmas.

Definition 37 *The relation $\twoheadrightarrow_{\beta\perp\mathsf{OA}}$ is defined as $(\twoheadrightarrow_{\beta\perp} \cup \rightarrow_{\mathsf{OA}})^*$.*

In order to avoid unnecessarily heavy notation we did not define $\twoheadrightarrow_{\beta\perp\mathsf{OA}}$ as a strongly converging reduction of arbitrary ordinal length, as customary in infinitary lambda calculus (Berarducci, 1996), (Kennaway et al., 1997), (Kennaway and de Vries, 2000). However, the reader may check that there is no loss of generality: any such arbitrary reduction would contain at most finitely many A, O-reduction steps, and the $\beta\perp$-reduction sequences in between can be compressed to $\beta\perp$-reductions of length at most ω.

We will prove confluence of strongly convergent reductions in $\Lambda^{\infty}_{\perp\mathsf{OA}}$ along the same lines as we proved confluence of finite reductions in $\Lambda_{\perp\mathsf{OA}}$.

Lemma 38 *(i) There is at most one* OA*-redex in a term belonging to* $\Lambda^{\infty}_{\perp\mathsf{OA}}$*.*
(ii) The relation $\to^*_{\mathsf{OA}}$ *commutes with the relation* $\twoheadrightarrow_{\beta\perp}$*.*

PROOF.

(i) By induction on $\Lambda^{\infty}_{\perp\mathsf{OA}}$.
(ii) Similar to the finite case considered in Lemma 34. After construction of the four base cases the proof proceeds now by induction on the ordinal length of $\twoheadrightarrow_{\beta\perp}$. The only interesting case is the limit ordinal ω: we construct

$$\begin{array}{ccccccc}
C[\mathsf{A}M] & \xrightarrow{\beta\perp} & C_1[\mathsf{A}M_1] & \xrightarrow{\beta\perp} & C_2[\mathsf{A}M_2] & \cdots\cdots & C_\omega[\mathsf{A}M_\omega] \\
{\scriptstyle\mathsf{OA}}\downarrow & & \downarrow{\scriptstyle\mathsf{OA}} & & \downarrow{\scriptstyle\mathsf{OA}} & & \downarrow{\scriptstyle\mathsf{OA}} \\
C[N] & \xrightarrow[\beta\perp]{} & C_1[N_1] & \xrightarrow[\beta\perp]{} & C_2[N_2] & \cdots\cdots & C_\omega[N_\omega]
\end{array}$$

Observe that the depth of the occurrences of A in the terms on the top row becomes fixed after a while. If that were not the case, then by the strongly convergence property there would be no A present in the limit. Now it is routine to verify that the reduction in the bottom row inherits the strongly convergence property of the reduction in the top row.

Theorem 39 *The relation* $\twoheadrightarrow_{\beta\perp\mathsf{OA}}$ *is confluent in* $\Lambda^{\infty}_{\perp\mathsf{OA}}$*:*

$$\begin{array}{ccc}
X & \xrightarrow{\beta\perp\mathsf{OA}}\!\!\!\!\rightarrow & Y_1 \\
{\scriptstyle\beta\perp\mathsf{OA}}\downarrow\!\!\downarrow & & \vdots\,{\scriptstyle\beta\perp\mathsf{OA}} \\
Y_2 & \cdots\underset{\beta\perp\mathsf{OA}}{\cdots}\twoheadrightarrow & Z
\end{array}$$

PROOF. Similar to the proof of Theorem 35 using the Hindley-Rosen Lemma, Theorem 11(iii) and Lemma 38. Notice that $\twoheadrightarrow_{\beta\perp\mathsf{OA}}$ is by definition $(\twoheadrightarrow_{\beta\perp} \cup \to_{\mathsf{OA}})^*$.

We have now the tools to conclude the unique normal form property for $\Lambda^{\infty}_{\perp\mathsf{OA}}$ from the unique normal form property for $\Lambda^{\infty}_{\perp}$.

Corollary 40 *For each term in* $\Lambda^{\infty}_{\perp\mathsf{OA}}$ *there is a unique normal form* N *such that* $M \twoheadrightarrow_{\beta\perp\mathsf{OA}} N$*.*

PROOF. Normalization follows from Lemma 26 and normalization of $\twoheadrightarrow_{\beta\bot}$ in $\Lambda^\infty_\bot$ (Theorem 11(iii)). Unicity follows from Theorem 39.

4.2 From Tree Equivalence to Observational Equivalence

We have now all the machinery to pull the rabbit out of the hat. First we will extend the definition of Berarducci tree from terms in $\Lambda^\infty_\bot$ to terms in $\Lambda^\infty_{\bot\mathsf{OA}}$. We will show the correspondence with the unique normal forms. We will conclude with a proof that Berarducci tree equality in $\Lambda^\infty_{\bot\mathsf{OA}}$ implies observational equivalence in $\Lambda^\infty_{\bot\mathsf{OA}}$.

Definition 41 *The* Berarducci tree $\mathcal{B}e\mathcal{T} : \Lambda^\infty_{\bot\mathsf{OA}} \to \Lambda^\infty_\bot$ *is defined by corecursion on* $\Lambda^\infty_{\bot\mathsf{OA}}$ *as follows:*

(i) if $X \to^*_{\beta\mathsf{OA}} x$ *then* $\mathcal{B}e\mathcal{T}(X) = x$;

(ii) if $X \to^*_{\beta\mathsf{OA}} \lambda x.M$ *then* $\mathcal{B}e\mathcal{T}(X) = \begin{array}{c}\lambda x\\ \mid\\ \mathcal{B}e\mathcal{T}(M)\end{array}$;

(iii) if $X \to^*_{\beta\mathsf{OA}} MN$ *and* M *is a* $\beta\mathsf{OA}$*-zero term then*

$$\mathcal{B}e\mathcal{T}(X) = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \mathcal{B}e\mathcal{T}(M) & & \mathcal{B}e\mathcal{T}(N)\end{array};$$

(iv) otherwise (exactly when X *is* $\beta\mathsf{OA}$*-rootactive),* $\mathcal{B}e\mathcal{T}(X) = \bot$.

Note that this definition does not need to consider clauses for O, A because of confluence of $\to_{\beta\mathsf{OA}}$ and Lemma 26.

Theorem 42 *Let* $X \in \Lambda^\infty_{\bot\mathsf{OA}}$.

(i) $\mathcal{B}e\mathcal{T}(X)$ *is in normal form;*
(ii) $X \twoheadrightarrow_{\beta\bot\mathsf{OA}} \mathcal{B}e\mathcal{T}(X)$;
(iii) $\mathcal{B}e\mathcal{T}(X)$ *is the unique normal form of* X.

PROOF.

(i) Suppose that $\mathcal{B}e\mathcal{T}(X)$ is not in normal form. Then a subtree of $\mathcal{B}e\mathcal{T}(X)$ contains a β-redex of the form $\mathcal{B}e\mathcal{T}(M)\mathcal{B}e\mathcal{T}(N)$. But M is a $\beta\mathsf{OA}$-zero term. A contradiction.
(ii) We consider the strongly convergent reduction sequence obtained by the depth-first outermost strategy [12]. The limit of this sequence satisfies the

[12] The depth-first outermost strategy reduces at each step the leftmost redex with minimal depth. Notice that this strategy applied to XY, where X is a $\beta\mathsf{OA}$-zero term and it has an infinite normal form and Y can be reduced, does always reduce

conditions of the definition of $\mathcal{B}e\mathcal{T}(X)$. By the coinduction principle, this limit is $\mathcal{B}e\mathcal{T}(X)$.

(iii) It follows from the previous parts and Corollary 40.

Corollary 43 *For all terms $X \in \Lambda^{\infty}_{\perp\mathsf{OA}}$ and context $C[\,] \in \Lambda^{\infty}_{\perp\mathsf{OA}}$ it holds that $\mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(X)]) = \mathcal{B}e\mathcal{T}(C[X])$.*

PROOF. Theorem 42(iii) gives the following diagram:

$$\begin{tikzcd}
C[X] \arrow[r, twoheadrightarrow, "\beta\perp\mathsf{OA}"] \arrow[d, twoheadrightarrow, "\beta\perp\mathsf{OA}"'] & C[\mathcal{B}e\mathcal{T}(X)] \arrow[d, twoheadrightarrow, "\beta\perp\mathsf{OA}"] \\
\mathcal{B}e\mathcal{T}(C[X]) \arrow[r, equal] & \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(X)])
\end{tikzcd}$$

Finally we can prove that Berarducci tree equality in Λ_{OA} and $\Lambda^{\infty}_{\perp\mathsf{OA}}$ implies observational equivalence respectively in Λ_{OA} and $\Lambda^{\infty}_{\perp\mathsf{OA}}$.

Theorem 44 *(i) For all $X, Y \in \Lambda_{\mathsf{OA}}$, $\mathcal{B}e\mathcal{T}(X) = \mathcal{B}e\mathcal{T}(Y)$ implies*

$$\forall C[\,] \in \Lambda_{\mathsf{OA}}.\ C[M] \in R_{\mathsf{OA}} \Leftrightarrow C[N] \in R_{\mathsf{OA}}.$$

(ii) For all $X, Y \in \Lambda^{\infty}_{\perp\mathsf{OA}}$, $\mathcal{B}e\mathcal{T}(X) = \mathcal{B}e\mathcal{T}(Y)$ implies

$$\forall C[\,] \in \Lambda^{\infty}_{\perp\mathsf{OA}}.\ C[X] \in \mathcal{R}^{\infty}_{\perp\mathsf{OA}} \Leftrightarrow C[Y] \in \mathcal{R}^{\infty}_{\perp\mathsf{OA}}.$$

PROOF. We prove (ii) since (i) is a particular case of (ii). Let X, Y be terms in $\Lambda^{\infty}_{\perp\mathsf{OA}}$. Suppose $\mathcal{B}e\mathcal{T}(X) = \mathcal{B}e\mathcal{T}(Y)$. Let $C[\,]$ be a context in $\Lambda^{\infty}_{\perp\mathsf{OA}}$. Using the previous corollary we get:

$$\begin{aligned}
\mathcal{B}e\mathcal{T}(C[X]) &= \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(X)]) \\
&= \mathcal{B}e\mathcal{T}(C[\mathcal{B}e\mathcal{T}(Y)]) \\
&= \mathcal{B}e\mathcal{T}(C[Y]).
\end{aligned}$$

Suppose $C[X] \in \mathcal{R}^{\infty}_{\perp\mathsf{OA}}$. Then $\mathcal{B}e\mathcal{T}(C[X]) = \perp$. Hence also $\mathcal{B}e\mathcal{T}(C[Y]) = \perp$. And so we find that $C[Y] \in \mathcal{R}^{\infty}_{\perp\mathsf{OA}}$. We conclude that X and Y are observationally equivalent.

Y after a finite number of steps. This is because if n is the minimal depth of redexes in Y, there is always an integer m such that if the depth-first outermost strategy applied to X after m reduction steps gives X', then the minimal depth of redexes in X' is greater than n.

Remark 45 Theorem 44 cannot be proved using approximants as for Böhm (Dezani-Ciancaglini et al., 1998b) or Lévy-Longo trees (Boudol and Laneve, 1996). This is because application is not continuous with respect to the Berarducci tree topology (see also (Berarducci and Dezani-Ciancaglini, 1999)). For example, take the context $C[\,] = [\,]\mathbf{I}$ and the directed set $X = \{\bot, \lambda x.\bot\}$. Clearly, $\bot = C[\bigsqcup X] \neq \bigsqcup C[X] = \bot\mathbf{I}$. Application is, a fortiori, not monotonic. E.g. $\bot \sqsubset \lambda x.\bot$, but $C[\bot] = \bot\mathbf{I} \sqsupset \bot = C[\lambda x.\bot]$.

5 Observational Equivalence implies Tree Equality

In this section we will prove that observational equivalence of terms in $\Lambda^\infty_{\bot\mathsf{OA}}$ with respect to the extended calculi Λ_{OA} implies equality of Berarducci trees. The proof will be a variant of the Böhm out technique (Barendregt, 1984) defined for Böhm trees.

Some terminology first. The *label at the root* of a tree T is denoted by $\mathsf{root}(T)$ and defined by cases:

$$\mathsf{root}(x) = x,\ \mathsf{root}\left(\begin{array}{c}\lambda x\\ |\\ T\end{array}\right) = \lambda x,\ \mathsf{root}\left(\begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ T_1 & & T_2\end{array}\right) = @,\text{ and } \mathsf{root}(\bot) = \bot.$$

Like in Definition 10.4.6 of (Barendregt, 1984) we will say that an occurrence is useful to discriminate between two Berarducci trees if the labeled nodes in all proper prefixes of the occurrence are identical, while the labeled nodes at the end of the occurrence are different.

Definition 46 *An occurrence u is useful for two trees T, T' if* $\mathsf{root}(T|_v) = \mathsf{root}(T'|_v)$ *for all $v < u$, but* $\mathsf{root}(T|_u) \neq \mathsf{root}(T'|_u)$.

We will use substitutions that map any variable in $\Lambda^\infty_\bot$ to a term in $\{\Omega, \Omega\Omega\}$. More precisely we will consider the substitution σ_Ω defined by

$$\sigma_\Omega(x) = \Omega \text{ for all variables } x$$

and the substitutions σ^x_Ω, one for each variable x, defined by

$$\sigma^x_\Omega(y) = \begin{cases}\Omega\Omega & \text{if } x = y\\ \Omega & \text{otherwise.}\end{cases}$$

Lemma 47 *Let $M \in \Lambda^\infty_\bot$ be a β-zero term and let σ be the substitution σ_Ω or σ^x_Ω for some fixed x. Then the substitution instance M^σ is a closed β-zero term.*

PROOF. By definition of σ, M^σ is a closed term. Suppose towards a contradiction that M^σ β-reduces to an abstraction. Then either M β-reduces to an abstraction or to a term of the shape $yN_1 \dots N_n$ for some variable y. By hypothesis M is a β-zero term and so it cannot β-reduce to an abstraction. Hence M β-reduces to $yN_1 \dots N_n$. This implies M^σ β-reduces to $\Omega N_1^\sigma \dots N_n^\sigma$ or to $\Omega\Omega N_1^\sigma \dots N_n^\sigma$, which are both closed β-zero terms.

Theorem 48 *(i) For all $X, Y \in \Lambda_{\mathsf{OA}}$ it holds that*

$$\forall C[\,] \in \Lambda_{\mathsf{OA}}\ C[X] \in R_{\mathsf{OA}} \Leftrightarrow C[Y] \in R_{\mathsf{OA}} \quad \Rightarrow \quad \mathcal{B}e\mathcal{T}(X) = \mathcal{B}e\mathcal{T}(Y).$$

(ii) For all $X, Y \in \Lambda^\infty_{\perp\mathsf{OA}}$ it holds that

$$\forall C[\,] \in \Lambda^\infty_{\perp\mathsf{OA}}\ C[X] \in R^\infty_{\perp\mathsf{OA}} \Leftrightarrow C[Y] \in R^\infty_{\perp\mathsf{OA}} \quad \Rightarrow \quad \mathcal{B}e\mathcal{T}(X) = \mathcal{B}e\mathcal{T}(Y).$$

PROOF. The proof of (i) and (ii) is essentially the same, since we consider only the Berarducci trees of X and Y which in both cases belong to $\Lambda^\infty_\perp$. So we only show (i). The proof will be by contraposition.

Let X, Y be terms in Λ_{OA} such that $\mathcal{B}e\mathcal{T}(X) \neq \mathcal{B}e\mathcal{T}(Y)$. Then there exists an occurrence u that is useful for $\mathcal{B}e\mathcal{T}(X)$ and $\mathcal{B}e\mathcal{T}(Y)$. Depending on what label we see at the root of $\mathcal{B}e\mathcal{T}(X)|_u$ and $\mathcal{B}e\mathcal{T}(Y)|_u$, we define a substitution σ as follows:

- If $\mathcal{B}e\mathcal{T}(X)|_u = x$ and $\mathcal{B}e\mathcal{T}(Y)|_u = y$, let σ be σ^x_Ω.
- If $\mathcal{B}e\mathcal{T}(X)|_u = x$ and $\mathcal{B}e\mathcal{T}(Y)|_u = \perp$ or conversely, let σ be σ^x_Ω.
- In all other cases let σ be σ_Ω.

By induction on the length of u we will define a context $C[\,] \in \Lambda_{\mathsf{OA}}$ that can discriminate X and Y with respect to σ in the sense that either $C[X^\sigma] \in R_{\mathsf{OA}}$ and $C[Y^\sigma] \notin R_{\mathsf{OA}}$, or vice versa.

Base case: $u = \langle\rangle$.

- If $\mathcal{B}e\mathcal{T}(X)$ or $\mathcal{B}e\mathcal{T}(Y)$ is a leaf, then we choose $C[\,] = [\,]$ as context to discriminate X and Y with respect to σ.
  We have four sub-cases:
  · if $\mathcal{B}e\mathcal{T}(X) = x$ and $\mathcal{B}e\mathcal{T}(Y) = y$ then $X^\sigma \to^*_{\beta\mathsf{OA}} \Omega\Omega$ and $Y^\sigma \to^*_{\beta\mathsf{OA}} \Omega$;
  · if $\mathcal{B}e\mathcal{T}(X) = x$ and $\mathcal{B}e\mathcal{T}(Y) = \perp$ (or vice versa) then $X^\sigma \to^*_{\beta\mathsf{OA}} \Omega\Omega \notin R_{\mathsf{OA}}$ and $Y^\sigma \in R_{\mathsf{OA}}$ (or vice versa);
  · if $\mathcal{B}e\mathcal{T}(X)$ is not a leaf and $\mathcal{B}e\mathcal{T}(Y) = x$ (or vice versa) then $X^\sigma \notin R_{\mathsf{OA}}$ and $Y^\sigma \to^*_{\beta\mathsf{OA}} \Omega \in R_{\mathsf{OA}}$ (or vice versa);
  · if $\mathcal{B}e\mathcal{T}(X)$ is not a leaf and $\mathcal{B}e\mathcal{T}(Y) = \perp$ (or vice versa) then $X^\sigma \notin R_{\mathsf{OA}}$ and $Y^\sigma \in R_{\mathsf{OA}}$ (or vice versa).

– If $\mathcal{B}e\mathcal{T}(X) = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \mathcal{B}e\mathcal{T}(M_1) & & \mathcal{B}e\mathcal{T}(M_2) \end{array}$ [13] with $X \to^*_{\beta\mathsf{OA}} M_1M_2$

and $\mathcal{B}e\mathcal{T}(Y) = \begin{array}{c} \lambda x \\ | \\ \mathcal{B}e\mathcal{T}(N_1) \end{array}$ with $X \to^*_{\beta\mathsf{OA}} \lambda x.N_1$ (or vice versa),

then we choose $C[\,] = \mathsf{O}[\,]\Omega$.
By the shape of $\mathcal{B}e\mathcal{T}(X)$ it follows that M_1 is a β-zero term. Hence M_1^σ is a closed β-zero term by Lemma 47. ¿From this fact and because $C[X^\sigma] \to^*_{\beta\mathsf{OA}} \mathsf{O}(M_1^\sigma M_2^\sigma)\Omega \to_{\mathsf{OA}} M_1^\sigma\Omega$, we find that is $C[X^\sigma]$ is $\beta\mathsf{OA}$-rootstable. On the other hand we find that $C[Y^\sigma]$ is $\beta\mathsf{OA}$-rootactive, because $C[Y^\sigma] \to^*_{\beta\mathsf{OA}} \mathsf{O}(\lambda x.N_1)^\sigma\Omega \to_{\mathsf{OA}} \mathbf{I}\Omega \to_\beta \Omega$.

Induction step: $u = i \cdot v$.

– Suppose $\mathcal{B}e\mathcal{T}(X) = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \mathcal{B}e\mathcal{T}(M_1) & & \mathcal{B}e\mathcal{T}(M_2) \end{array}$ with $X \to^*_{\beta\mathsf{OA}} M_1M_2$

and $\mathcal{B}e\mathcal{T}(Y) = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \mathcal{B}e\mathcal{T}(N_1) & & \mathcal{B}e\mathcal{T}(N_2) \end{array}$ with $Y \to^*_{\beta\mathsf{OA}} N_1N_2$.

We have two sub-cases:

· If $i = 1$ then by the induction hypothesis we have a context $C'[\,]$ that discriminates M_1, N_1 with respect to σ. Then we define $C[\,] = C'[\mathsf{O}[\,]]$. As in the base case we get that M_1^σ is a closed β-zero term. Now clearly $C[X^\sigma] \to^*_{\beta\mathsf{OA}} C'[\mathsf{O}(M_1^\sigma M_2^\sigma)] \to_{\mathsf{OA}} C'[M_1^\sigma]$ and similarly $C[Y^\sigma] \to^*_{\beta\mathsf{OA}} C'[\mathsf{O}(N_1^\sigma N_2^\sigma)] \to_{\mathsf{OA}} C'[N_1^\sigma]$. Hence by induction $C[\,]$ discriminates X and Y with respect to σ.

· If on the other hand $i = 2$, then by the induction hypothesis there is a context $C'[\,]$ that discriminates M_2 and N_2 with respect to σ. We now choose $C[\,] = C'[\mathsf{A}[\,]]$ to discriminate X and Y with respect to σ. The proof proceeds as before. Again M_1 is a β-zero term, and M_1^σ is a closed β-zero term. So we can calculate that $C[X^\sigma] \to^*_{\beta\mathsf{OA}} C'[\mathsf{A}(M_1^\sigma M_2^\sigma)] \to_{\mathsf{OA}} C'[M_2^\sigma]$ and similarly we see that $C[Y^\sigma] \to^*_{\beta\mathsf{OA}} C'[\mathsf{A}(N_1^\sigma N_2^\sigma)] \to_{\mathsf{OA}} C'[N_2^\sigma]$. Hence by induction $C[\,]$ discriminates X and Y with respect to σ.

– Suppose $\mathcal{B}e\mathcal{T}(X) = \begin{array}{c} \lambda x \\ | \\ \mathcal{B}e\mathcal{T}(M_1) \end{array}$ with $X \to^*_{\beta\mathsf{OA}} \lambda x.M_1$

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[13] Notice that $\mathcal{B}e\mathcal{T}(X) = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \mathcal{B}e\mathcal{T}(M_1) & & \mathcal{B}e\mathcal{T}(M_2) \end{array}$ does not imply $X \to^*_{\beta\mathsf{OA}} M_1M_2$, since, for example, $\mathcal{B}e\mathcal{T}(\mathbf{I}(\Omega\Omega)) = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \mathcal{B}e\mathcal{T}(\mathbf{I}\Omega) & & \mathcal{B}e\mathcal{T}(\mathbf{I}\Omega) \end{array} = \begin{array}{ccc} & @ & \\ \diagup & & \diagdown \\ \bot & & \bot \end{array}$.

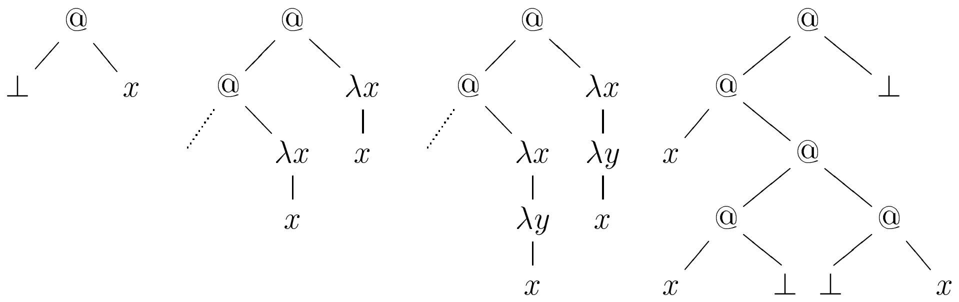

Fig. 3. Berarducci trees of $\Omega x, \Omega_{\mathbf{I}}$, $\Omega_{\mathbf{K}}$, and $x(x\Omega(\Omega x))\Omega$.

and $\mathcal{B}e\mathcal{T}(Y) = \begin{array}{c}\lambda x\\ \mid \\ \mathcal{B}e\mathcal{T}(N_1)\end{array}$ with $Y \to^*_{\beta\mathsf{OA}} \lambda x.N_1$.
Then $i = 0$. Let $C'[\ ]$ be the context that by induction hypothesis discriminates M_1 and N_1. We now choose $C[\ ] = C'[[\ ]\sigma(x)]$. We observe that $C[X^\sigma] \to^*_{\beta\mathsf{OA}} C'[(\lambda x.M_1)^\sigma \sigma(x)] \to_\beta C'[M_1^\sigma]$ and similarly we see that $C[Y^\sigma] \to^*_{\beta\mathsf{OA}} C'[(\lambda x.N_1)^\sigma \sigma(x)] \to_\beta C'[N_1^\sigma]$. Hence, by induction, $C[\ ]$ discriminates X and Y with respect to σ.

Recapitulating, given the two terms X, Y in Λ_{OA} and an occurrence u that is useful to discriminate their Berarducci trees, we have constructed a context $C'[\ ]$ together with a substitution σ able to discriminate X and Y. To finish off the proof we will now build a context from these two ingredients that can discriminate X and Y:

$$C[\ ] = C'[(\lambda x_1 ... x_n.[\ ])\sigma(x_1) \dots \sigma(x_n)],$$

where $x_1, \dots x_n$ is the set of free variables in X and Y.

Since $(\lambda x_1 ... x_n.X)\sigma(x_1) \dots \sigma(x_n)$ and $(\lambda x_1 ... x_n.Y)\sigma(x_1) \dots \sigma(x_n)$ are closed, we note that $C[X]$ and $C[Y]$ belong to Λ_{OA}. Now, because

$$C[X] = C'[(\lambda x_1 ... x_n.X)\sigma(x_1) \dots \sigma(x_n)] \to^*_{\beta\mathsf{OA}} C'[X^\sigma]$$

and similarly $C[Y] \to^*_{\beta\mathsf{OA}} C'[Y^\sigma]$ and by construction $C'[\ ]$ discriminates X, Y with respect to σ, we get that $C[X]$ is $\beta\mathsf{OA}$-rootactive and $C[Y]$ is not, or vice versa.

Example 49 *The Berarducci trees of some terms considered in this example are shown in Figure 3.*

(i) When $M = \Omega$, $N = \Omega\Omega$ and $u = \langle\rangle$ the above procedure gives us the empty context as a discriminating context for M and N.
(ii) If $M = \Omega$, $N = \Omega x$, and $u = \langle\rangle$, then we find that $C[\ ] = (\lambda x.[\ ])\Omega$ discriminates M and N.

(iii) *For* $M = \Omega x$, $N = \Omega y$ *and* $u = 2$ *we find that* $C[\ ] = \mathsf{A}((\lambda xy.[\ ])(\Omega\Omega)\Omega)$ *is a discriminating context.*

(iv) *Let* $M = \Omega_{\mathbf{I}}$, $N = \Omega_{\mathbf{K}}$, *and* $u = 2 \cdot 0$. *The discriminating context we obtain is* $C[\ ] = \mathsf{A}[\ ]\Omega$.

(v) *In case of* $M = x(x\Omega(\Omega x))\Omega$, $N = y(y\Omega(\Omega y))\Omega$, *and* $u = 1 \cdot 2 \cdot 2 \cdot 2$, *a discriminating context is* $C[\ ] = \mathsf{A}(\mathsf{A}(\mathsf{A}(\mathsf{O}((\lambda xy.[\ ])(\Omega\Omega)\Omega))))$.

This last case shows the power of the constants O, A. One problem in constructing such discriminating contexts is that different occurrences of the same variable may have to be used to select different arguments. This problem was solved in the original algorithm of Böhm by using suitable combinators which equate η-convertible terms (see Section 10.4 of (Barendregt, 1984)) and in (Sangiorgi, 1994), (Dezani-Ciancaglini et al., 1999), (Dezani-Ciancaglini et al., 1998b) by allowing a non-deterministic choice operator. In all these cases the trick is to replace different occurrences of the same variable by different terms. Instead, in the above algorithm for Berarducci trees the selection is performed by the two constants O and A while the variables always get substituted by Ω or $\Omega\Omega$.

By Theorems 48 and 44, Berarducci tree equality of terms (possibly non-pure and/or infinite) coincides with observational equivalence. So the Berarducci trees build a fully abstract model of the (infinitary) lambda calculus extended with the constants O and A.

6 Conclusions

In (Sangiorgi, 1994) Sangiorgi proves that by adding well-formed operators to pure lambda calculus we cannot discriminate more than Lévy-Longo trees do. As a matter of fact, our operators O, A are not well-formed according to the Groote-Vaandrager format allowed in (Sangiorgi, 1994). The reason is that this format does not allow a premise asking for a term to be a closed β-zero term. In this respect our development completely agrees with that of Sangiorgi.

Looking back at the present work and the related papers (Dezani-Ciancaglini et al., 1998b) and (Dezani-Ciancaglini et al., 1999) that define extensions of pure lambda calculus that can internally discriminate as respectively Böhm trees and Lévy-Longo trees do, then one can wonder to what extent the chosen discriminating extensions actually depend on the nature of the problems dealt with. For instance it is not clear whether there are extensions of lambda calculus completely different from the present one and which internally discriminate as Berarducci trees do: we are tempted to conjecture that the extension with O, A is minimal in the sense that any other extension with the same discriminatory power contains translations of O, A and their rewrite rules.

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